

QUADRATIC TWISTS OF RIGID CALABI-YAU THREEFOLDS OVER \mathbb{Q}

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ABSTRACT. We consider rigid Calabi–Yau threefolds defined over \mathbb{Q} and the question of whether they admit quadratic twists. We give a precise geometric definition of the notion of a quadratic twists in this setting.

Every rigid Calabi–Yau threefold over \mathbb{Q} is modular so there is attached to it a certain newform of weight 4 on some $\Gamma_0(N)$. We show that quadratic twisting of a threefold corresponds to twisting the attached newform by quadratic characters and illustrate with a number of obvious and not so obvious examples.

The question is motivated by the deeper question of which newforms of weight 4 on some $\Gamma_0(N)$ and integral Fourier coefficients arise from rigid Calabi–Yau threefolds defined over \mathbb{Q} (a geometric realization problem).

1. INTRODUCTION

Suppose X is a rigid Calabi–Yau threefold defined over \mathbb{Q} . As Gouvêa and Yui observe in [9] (see also [4], [5]), it follows from work of Khare and Winterberger that X is modular: The L -series of X coincides with the L -series of a certain newform f of weight 4 on some $\Gamma_0(N)$. Alternatively, there is a newform f with integer coefficients such that, for any prime ℓ , the ℓ -adic representation of $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H^3(\bar{X}, \mathbb{Q}_{\ell})$ is isomorphic to the ℓ -adic representation of $G_{\mathbb{Q}}$ attached to f .

Very little seems to be known about the form f . Notably, the relation between the conductor N and the geometry of X still seems to be poorly understood. (See the discussions and conjectures of section 6.4 of [13], as well as the paper [3].)

Another unresolved and probably very hard question is the following. The form f above obviously has integral Fourier coefficients. Can one conversely characterize the newforms of weight 4 on some $\Gamma_0(N)$ with integral coefficients that arise from rigid Calabi–Yau threefolds over \mathbb{Q} ? Do all such forms arise from Calabi–Yau threefolds? (This is a kind of the geometric realization problem. See [6] for the case of “singular” K3 surfaces and forms of weight 3.)

A very weak version of this question is the topic of this paper: Given a rigid Calabi–Yau threefold X with form f as above, for any non-square rational number d there is a twist f_d of f by the quadratic character corresponding to the quadratic extension $K = \mathbb{Q}(\sqrt{d})$ over \mathbb{Q} . This f_d is again of the above form and so we can ask whether f_d arises from a rigid Calabi–Yau threefold X_d over \mathbb{Q} . This will be the case whenever X admits a quadratic twist by d in the sense we discuss next.

2010 *Mathematics Subject Classification.* 14J32, 11F80, 11F11.

This work was partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC), and by The Danish Council for Independent Research.

2. QUADRATIC TWISTS OF RIGID CALABI–YAU THREEFOLDS

Let X be a rigid Calabi–Yau threefold defined over \mathbb{Q} . Suppose that $d \in \mathbb{Q}^\times$ is a squarefree integer, let $K := \mathbb{Q}(\sqrt{d})$, and let σ be the non-trivial automorphism of K/\mathbb{Q} . We say that a rigid Calabi–Yau threefold X_d defined over \mathbb{Q} is a *twist* of X by d if there exist:

- an involution ι of $X_{\mathbb{Q}}$ that acts as -1 on $H^3(\bar{X}, \mathbb{Q}_\ell)$ for some prime ℓ , and
- an isomorphism $\theta: (X_d)_K \xrightarrow{\cong} X_K$ defined over K

such that:

$$\theta^\sigma \circ \theta^{-1} = \iota.$$

Notice that the condition $\iota = \theta^\sigma \circ \theta^{-1}$ necessarily implies that the involution ι satisfies $\iota^\sigma = \iota$, i.e., that ι is defined over \mathbb{Q} . Conversely, given an involution ι on $X_{\mathbb{Q}}$, one can always find the isomorphism θ . One takes the quotient of $X_K = X_{\mathbb{Q}} \otimes K$ by $\iota \otimes \sigma$, checks that it is defined over \mathbb{Q} and that it is the twist X_d as above. This will become clear in the examples below: whenever we can find the appropriate involution we can also construct a twist.

Since X is a rigid Calabi–Yau threefold, $H^{3,0}(X)$ is one-dimensional so there is a unique (up to scalar) holomorphic 3-form Ω on X . The involution ι should act on Ω non-symplectically, sending it to $-\Omega$. Conversely, since X is rigid we have $h^{2,1}(X) = 0$, so if ι sends Ω to $-\Omega$ we see that ι acts as -1 on all of $H^3(\bar{X}, \mathbb{Q}_\ell)$. (Here the rigidity of X is used in an essential way.)

This is the method that we will primarily employ in the examples below to ensure this part of the condition on the involution ι .

One could envision relaxing the above definition in the direction of just requiring the existence of an algebraic correspondence between $(X_d)_K$ and X_K and still retain (a somewhat stronger version of) the theorem below. However, in the examples that we will give, we actually find isomorphisms in all cases and have hence chosen to work with the above definition.

The principles of proof of the following theorem should be well-known, but we provide the details because of lack of a precise reference.

Recall that, given a newform f of some weight and a non-square $d \in \mathbb{Q}$ there is a twist f_d of f by d which is again a newform of the same weight as f (but potentially at another level) and whose attached ℓ -adic Galois representation (for some prime ℓ and hence for all primes ℓ) is isomorphic to the ℓ -adic representation attached to f twisted by the quadratic character χ corresponding to K/\mathbb{Q} . If the Fourier coefficients of f and f_d are a_n and b_n , respectively, we have the relation $b_p = \chi(p)a_p$ for almost all primes p . In particular, since χ is quadratic, if f has coefficients in \mathbb{Z} then so does f_d .

Theorem 1. *In the above setting, suppose that the newform (of weight 4) attached to X is f . Then, if X_d is a twist by d of X the newform attached to X_d is f_d , the twist of f by the Dirichlet character χ corresponding to the quadratic extension $K = \mathbb{Q}(\sqrt{d})$ of \mathbb{Q} .*

If we keep all hypotheses above except possibly that ι acts as -1 on $H^3(\bar{X}, \mathbb{Q}_\ell)$, we can still deduce that the newform attached to X_d is either f or f_d .

Proof. Fix a prime number ℓ , and consider the ℓ -adic Galois representations ρ and ρ_d attached to X and X_d , respectively: these are given by the action of

$G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the 2-dimensional \mathbb{Q}_{ℓ} -vector spaces $V := H^3(\bar{X}, \mathbb{Q}_{\ell})$ and $V_d := H^3(\bar{X}_d, \mathbb{Q}_{\ell})$, respectively.

Now, the newform f attached to X is determined uniquely by the requirement that its attached ℓ -adic representation be isomorphic to ρ . Similarly, the newform attached to X_d is determined by the requirement that its attached ℓ -adic representation be isomorphic to ρ_d .

Since the ℓ -adic representation attached to f_d is isomorphic to the twist by χ of the one attached to f , we see that what we have to prove boils down to:

$$\rho_d \cong \rho \otimes \chi.$$

Put $N := G_K = \text{Gal}(\overline{\mathbb{Q}}/K)$ so that N is a normal subgroup of $G_{\mathbb{Q}}$. The existence of the isomorphism $\theta: (X_d)_K \cong X_K$ defined over K translates into the existence of a \mathbb{Q}_{ℓ} -linear isomorphism $V_d \rightarrow V$ commuting with the action of G_K . I.e., in matrix terms we have an invertible matrix A with:

$$\rho(n)A = A\rho_d(n) \quad \text{for all } n \in N.$$

In matrix terms the conjugate isomorphism θ^{σ} of V_d onto V is then given by the matrix

$$\rho(\sigma)A\rho_d(\sigma)^{-1};$$

notice that we have here viewed σ as an element of $G_{\mathbb{Q}}$ via choice of a representative; the expression $\rho(\sigma)A\rho_d(\sigma)^{-1}$ does not depend on this choice.

If now ι acts as -1 on $H^3(\bar{X}, \mathbb{Q}_{\ell})$ we can deduce that the matrix

$$\rho(\sigma)A\rho_d(\sigma)^{-1}A^{-1}$$

is a non-trivial involution.

Define the representation ρ' of $G_{\mathbb{Q}}$ by $\rho' := A^{-1}\rho A$ so that $\rho'(n) = \rho_d(n)$ for $n \in N$. Then, for arbitrary $g \in G_{\mathbb{Q}}$ and $n \in N$ we have

$$\begin{aligned} \rho_d(g)\rho'(n)\rho_d(g)^{-1} &= \rho_d(g)\rho_d(n)\rho_d(g)^{-1} \\ &= \rho_d(gn g^{-1}) = \rho'(gn g^{-1}) = \rho'(g)\rho'(n)\rho'(g)^{-1} \end{aligned}$$

so that

$$\rho'(g)^{-1}\rho_d(g)\rho'(n) = \rho'(n)\rho'(g)^{-1}\rho_d(g)$$

i.e., for any $g \in G_{\mathbb{Q}}$ the matrix $\rho'(g)^{-1}\rho_d(g)$ commutes with all matrices $\rho'(n)$, $n \in N$.

Now, suppose first that ρ (and hence ρ') is absolutely irreducible when restricted to N . In that case we deduce that $\rho'(g)^{-1}\rho_d(g)$ is a scalar matrix, say with diagonal entry $\mu(g)$. We have $\mu(n) = 1$ for $n \in N$ and see that $g \mapsto \mu(g)$ is in fact a character of $G_{\mathbb{Q}}$ factoring through $N = G_K$. So, either $\mu = 1$ or $\mu = \chi$.

If we had $\mu = 1$ we would have $A^{-1}\rho(g)A = \rho'(g) = \rho_d(g)$ for all $g \in G_{\mathbb{Q}}$ and so in particular the matrix

$$\rho(\sigma)A\rho_d(\sigma)^{-1}A^{-1}$$

would be trivial. As we noted above, this can not happen if ι acts as -1 on $H^3(\bar{X}, \mathbb{Q}_{\ell})$. Hence, in that case we must have $\mu = \chi$ and so $\rho_d = \rho' \otimes \chi \cong \rho \otimes \chi$, as desired.

Suppose now that ρ is not absolutely irreducible when restricted to G_K . The same is then true of ρ' and ρ_d . In this case it is known, cf. (4.4), (4.5) of [16], that ρ' is induced from the ℓ -adic representation ψ attached to a Grössencharacter over K : $\rho' = \text{Ind}_{K/\mathbb{Q}}(\psi)$, and ρ'_{G_K} splits up as the sum of the two characters ψ and ψ^{σ} .

Notice that $\text{Ind}_{K/\mathbb{Q}}(\psi) = \text{Ind}_{K/\mathbb{Q}}(\psi^\sigma)$. Since ρ' and ρ_d have the same restriction to G_K we may then conclude that in fact $\rho' = \rho_d$ as representations of $G_{\mathbb{Q}}$, and hence that the newform attached to X_d is f . Furthermore, the matrix $\rho(\sigma)A\rho_d(\sigma)^{-1}A^{-1}$ must then be trivial, and so we see that this case in fact does not materialize if ι acts as -1 on $H^3(\bar{X}, \mathbb{Q}_\ell)$. \square

Remark 1. *What we have proved, in fact, is that if ι is nontrivial the Galois representation on the middle cohomology of X_d is isomorphic to the tensor product of the representation on the middle cohomology of X and the one-dimensional Galois representation corresponding to $K = \mathbb{Q}(\sqrt{d})$. For rigid Calabi–Yau manifolds, we know these representations correspond to modular forms, but the question can, of course, be asked without knowing anything about modularity. We are grateful to the referee for pointing this out to us.*

2.1. Easy examples of twists. The standard, simple example of twisting is of course for an elliptic curve E over \mathbb{Q} , say given by a Weierstrass equation $y^2 = x^3 + ax^2 + bx + c$. The twisted curve E_d is then given by the equation $dy^2 = x^3 + ax^2 + bx + c$ with the isomorphism $\theta : E_d \rightarrow E$ defined over $K = \mathbb{Q}(\sqrt{d})$ by $\theta(x, y) = (x, \sqrt{d}y)$. The corresponding involution ι is $(x, y) \mapsto (x, -y)$. It is clear that ι sends the holomorphic 1-form $\Omega = \frac{dx}{y}$ to $-\Omega$.

For a number of rigid Calabi–Yau threefolds over \mathbb{Q} we can display twists by essentially the same method: Consider for examples the various cases of double octic Calabi–Yau threefolds over \mathbb{Q} (see [13] for instance for a good overview). They are defined as hypersurfaces of the form

$$y^2 = f_8(x_1, x_2, x_3, x_4)$$

where f_8 is a degree 8 homogeneous polynomial. As in the case of elliptic curves, we have an obvious twist given by

$$dy^2 = f_8(x_1, x_2, x_3, x_4).$$

The corresponding involution is of course again given by $y \mapsto -y$. Again it is clear that ι sends the holomorphic 3-form

$$\Omega = \frac{\sum_{i=1}^4 (-1)^i x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_4}{y}$$

to $-\Omega$.

This is also completely analogous to the case certain modular double sextic $K3$ surfaces, see [14]. These have form

$$w^2 = f_6(x, y, z)$$

where f_6 is a projective smooth curve of degree 6. As above, we get a twist of this surface (in the sense analogous to Theorem 1) via the twisted equation

$$dw^2 = f_6(x, y, z)$$

for a non-square rational number d . The involution is again given by $w \mapsto -w$. The holomorphic 2-form

$$\Omega = \frac{z dx \wedge dy - x dy \wedge dz + y dx \wedge dz}{w}$$

is sent by ι to $-\Omega$.

2.2. Self-fiber products of rational elliptic surfaces with section and their twists. Slightly more complicated examples arise in connection with the rigid Calabi–Yau threefolds studied by H. Verrill in the appendix to [21]: She determined the L -series via the point counting method for the six isomorphism classes of rigid Calabi–Yau threefolds constructed as self-fiber products of rational elliptic surfaces with section by Schoen [18]. Along the way, she discussed twists by quadratic characters.

These six rigid Calabi–Yau threefolds over \mathbb{Q} are defined as follows: Start with semi-stable families of elliptic curves $\pi : \mathcal{Y} \rightarrow \mathbb{P}^1$, i.e., \mathcal{Y} is a smooth surface and the singular fibers have type I_m . Beauville [1] gave a complete list of these families. These are realized as the resolutions of singular surfaces $\bar{\mathcal{Y}} \subset \mathbb{P}^2 \times \mathbb{P}^1$ given by the following equations:

#	Equation for $\bar{\mathcal{Y}}$	
<i>I</i>	$(x^3 + y^3 + z^3)\mu$	$= \lambda xyz$
<i>II</i>	$x(x^2 + z^2 + 2zy)\mu$	$= \lambda(x^2 - y^2)z$
<i>III</i>	$x(x - z)(y - z)\mu$	$= \lambda(x - y)yz$
<i>IV</i>	$(x + y + z)(xy + yz + zx)\mu$	$= \lambda xyz$
<i>V</i>	$(x + y)(xy - z^2)\mu$	$= \lambda xyz$
<i>VI</i>	$(x^2y + y^2z + z^2x)\mu$	$= \lambda xyz$

The fibration $\bar{\pi} : \bar{\mathcal{Y}} \rightarrow \mathbb{P}^1$ is given by projecting to \mathbb{P}^1 , and \mathcal{Y} is obtained by resolving $\bar{\mathcal{Y}}$. Now take the self-fiber product $\mathcal{Y} \times_{\mathbb{P}^1} \mathcal{Y}$. Schoen [18] shows that a small resolution exists and that the resulting smooth variety X is a rigid Calabi–Yau threefold defined over \mathbb{Q} . Thus, in each case there is a newform of weight 4 attached to f . In each case, the form was identified by Verrill via determination of the L -series of X (point counting.) Here is the table of newforms from Verrill.

#	Newform	modular group	level
<i>I</i>	$\eta(q^3)^8$	$\Gamma(3)$	9
<i>II</i>	$\eta(q^2)^4\eta(q^4)^4$	$\Gamma_1(4) \cap \Gamma(2)$	8
<i>III</i>	$\eta(q)^4\eta(q^5)^4$	$\Gamma_1(5)$	5
<i>IV</i>	$\eta(q)^2\eta(q^2)^2\eta(q^3)^2\eta(q^6)^2$	$\Gamma_1(6)$	6
<i>V</i>	$\eta(q^4)^{16}\eta(q^8)^{-4}\eta(q^2)^{-4}$	$\Gamma_0(8) \cap \Gamma_1(4)$	16
<i>VI</i>	$\eta(q^3)^8$	$\Gamma_0(9) \cap \Gamma_1(3)$	9

In each case, one can display a twist X_d of X so that X_d corresponds to twisting the newform by the quadratic character belonging to $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. Consider for instance type *V* above. Given a non-square $d \in \mathbb{Q}^\times$ let X_d be the variety arising from the equation

$$(x + y)(xy - dz^2)\mu = \lambda xyz$$

by a process analogous to the one leading to X above.

Then we have an isomorphism $\theta : X_d \rightarrow X$ defined over $\mathbb{Q}(\sqrt{d})$ and given by

$$((x : y : z), (\mu : \lambda)) \mapsto ((\sqrt{d}x : \sqrt{d}y : z), (\mu : \sqrt{d}\lambda)).$$

In the setup of Theorem 1, the involution ι is given by

$$\iota((x : y : z), (\mu : \lambda)) = ((-x : -y : z), (\mu : -\lambda)).$$

That X_d is a genuine twist of X , i.e., that the attached newform is f_d rather than f can be ascertained via point counting, cf. appendix in [21].

The other examples can be dealt with in similar fashions.

2.3. The Schoen quintic and its quadratic twists. As a more interesting test case, we consider the Schoen quintic

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 5x_0x_1x_2x_3x_4.$$

We write

$$f = f(x_0, x_1, x_2, x_3, x_4) = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5x_0x_1x_2x_3x_4 = 0.$$

This is a singular threefold with 125 nodes (ordinary double points) as only singularities, and a small resolution of singularities produces a rigid Calabi-Yau threefold X that is known, cf. [17], to be associated to a newform of weight 4 and level 25 (the modular form 25k4A1); see also [13], section 3.1.

We seek an involution ι of X that acts on $H^{3,0}(X)$ as multiplication by -1 . Since $H^{3,0}(X)$ is generated by a unique holomorphic 3-form Ω (up to scalar), ι should send Ω to $-\Omega$.

To determine the action of ι on Ω we can use either of the following two arguments:

According to Cox and Katz [2], especially section 2.3 and the formula (2.7) therein, Ω can be computed on the smooth part as

$$\Omega = \text{Res}\left(\frac{\omega}{f}\right)$$

where

$$\omega = \sum_{i=0}^4 (-1)^i x_i dx_0 \cdots \wedge \widehat{dx_i} \wedge \cdots dx_4$$

and where ‘Res’ denotes Poincaré residue.

Alternatively, it follows from Lemma 1 below that we have a holomorphic 3-form

$$\frac{dx_0 \wedge dx_1 \wedge dx_2}{\partial f / \partial x_3}.$$

on the Zariski open set where x_4 and $\partial f / \partial x_3$ are both non-vanishing, and that this extends to the Calabi-Yau threefold X .

Can one construct the requisite quadratic twists of the Schoen quintic? In Gouvêa and Yui [9] it was briefly asserted that quadratic twist indeed exists for the Schoen quintic. We now discuss details of this claim.

Proposition 1. *For any non-square $d \in \mathbb{Q}^\times$ the Schoen quintic has a twist by d . The corresponding involution ι defined over \mathbb{Q} is given explicitly on the coordinates by*

$$\iota : (x_0, x_1, x_2, x_3, x_4) \mapsto (x_1, x_0, x_2, x_3, x_4),$$

and sends $\Omega \in H^{3,0}(X)$ to $-\Omega$.

Proof. First, it is plain that ι sends the above Ω to $-\Omega$ (using any of the two descriptions of Ω .)

Put $u = x_0 + x_1$ and $v = x_0 - x_1$. Then the equation for the quintic equation can be written as a polynomial in u and v^2 as follows:

$$u^5 + 10u^3v^2 + 5uv^4 + 16(x_2^5 + x_3^5 + x_4^5) - 20(u^2 - v^2)x_2x_3x_4 = 0.$$

Replacing v by $\sqrt{d}v$, we obtain the following quintic equation:

$$(*) \quad u^5 + 10du^3v^2 + 5d^2uv^4 + 16(x_2^5 + x_3^5 + x_4^5) - 20(u^2 - dv^2)x_2x_3x_4 = 0,$$

and we see how to apply Theorem 1: The equation $(*)$ gives rise to a rigid Calabi–Yau threefold X_d defined over \mathbb{Q} . Then we have an isomorphism $\theta: X_d \rightarrow X$ defined over $\mathbb{Q}(\sqrt{d})$ and given by

$$(u, v, x_2, x_3, x_4) \mapsto (u, \sqrt{d}v, x_2, x_3, x_4)$$

so that $\theta^\sigma \circ \theta^{-1}$ is the involution given by $(u, v, x_2, x_3, x_4) = (u, -v, x_2, x_3, x_4)$. This is precisely the involution ι so the existence of the twist follows from Theorem 1. \square

2.4. Explicit description for a holomorphic 3-form for a complete intersection Calabi–Yau threefold. Before we go into further examples, we give an explicit description of a holomorphic 3-form for a complete intersection Calabi–Yau threefold, by the Griffiths residue theorem or its generalized version. We are grateful to Bert van Geemen for communicating to us the following lemma as well as its proof.

Lemma 1. *Let $Y = V(f_1, \dots, f_k)$ be a complete intersection in \mathbb{P}^n of dimension $d =: n - k$ where f_1, \dots, f_k are homogeneous equations in the homogeneous variables x_0, \dots, x_n . Assume that Y is a normal crossings divisor.*

Let $i_0 \in \{0, \dots, n\}$, let $I \subseteq \{0, \dots, n\} \setminus \{i_0\}$ have cardinality k , and consider

$$D_I := \det \left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{1 \leq i \leq k \\ j \in I}}$$

Then, on the Zariski open set where x_{i_0} and D_I are both non-vanishing, a holomorphic d -form is given by

$$\Omega = \frac{\bigwedge_{j \in \{0, \dots, n\} \setminus (\{i_0\} \cup I)} dx_j}{D_I}$$

If additionally Y has a crepant resolution X that is Calabi–Yau variety of dimension $\dim X \leq 3$, then Ω extends to all of X .

Let us remind that a ‘crepant resolution’ is one that does not change the canonical class, cf. [15], §2.

The Lemma applies to the Schoen quintic as well as the threefolds that we shall consider below because the singularities involved are ordinary double point in all cases.

The proof is given below in the appendix (section 5.)

2.5. Two rigid Calabi–Yau threefolds of Werner and van Geemen. Werner and van Geemen [20] constructed a number of examples of rigid Calabi–Yau threefolds over \mathbb{Q} . They are complete intersection Calabi–Yau threefolds.

We consider two of them. First, the rigid Calabi–Yau threefold denoted by \tilde{V}_{33} : Let $V_{33} \subset \mathbb{P}^5$ be the threefold defined by the system of equations

$$\begin{aligned} x_0^3 + x_1^3 + x_2^3 + x_3^3 &= 0 \\ x_2^3 + x_3^3 + x_4^3 + x_5^3 &= 0. \end{aligned}$$

V_{33} has 9 singularities, and let \tilde{V}_{33} be the blow up of V_{33} along its singular locus (big resolution). Then \tilde{V}_{33} is a rigid Calabi–Yau threefold over \mathbb{Q} , and it is modular with a corresponding newform f of weight 4 on $\Gamma_0(9)$:

$$f(q) = \eta(q^3)^8.$$

It is shown by Kimura [12] that if $E \subset \mathbb{P}^2$ is the curve defined by $x_0^3 + x_1^3 + x_2^3 = 0$, then there is a dominant rational map from E^3 to V_{33} of degree 3. Consequently, the L -series coincide. By Lemma 1, a holomorphic 3-form of V_{33} is given in affine coordinates by

$$\Omega = \frac{dx_2 \wedge dx_3 \wedge dx_4}{x_1^2 x_5^2}.$$

Proposition 2. *For any non-square $d \in \mathbb{Q}^\times$ the rigid Calabi–Yau threefold \tilde{V}_{33} has a twist $\tilde{V}_{33,d}$ by d . The corresponding involution ι is defined by permuting x_2 and x_3 .*

Proof. Put $u = x_2 + x_3$, $v = x_2 - x_3$. Then the equation for V_{33} can be expressed in terms of x_0, x_1, x_4, x_5 and u and v^2 :

$$\begin{aligned} 4x_0^3 + 4x_1^3 + u^3 + 3uv^2 &= 0 \\ u^3 + 3uv^2 + 4x_4^3 + 4x_5^3 &= 0. \end{aligned}$$

Replacing v by $\sqrt{d}v$ in this system we obtain a system of equations that gives rise to $\tilde{V}_{33,d}$. Applying Theorem 1 shows that $\tilde{V}_{33,d}$ is twist by d of \tilde{V}_{33} with the corresponding involution given by $v \mapsto -v$, i.e., by $(x_2, x_3) \mapsto (x_3, x_2)$.

The holomorphic 3-form Ω above clearly changes sign when x_2 and x_3 are interchanged. \square

Secondly, we can consider the rigid Calabi–Yau threefold denoted by \tilde{V}_{24} : Let $V_{24} \subset \mathbb{P}^5$ be the threefold defined by the equations:

$$\begin{aligned} x_0^2 + x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 &= 0 \\ x_0^4 + x_1^4 + x_2^4 - x_3^4 - x_4^4 - x_5^4 &= 0. \end{aligned}$$

Then V_{24} has 122 nodes (ordinary double points) as only singularities. Let \tilde{V}_{24} be the blow up of V_{24} along its singular locus (small resolution). Then \tilde{V}_{24} is a rigid Calabi–Yau threefold over \mathbb{Q} , and it is modular with a corresponding newform g which is the newform of weight 4 on $\Gamma_0(12)$.

Proposition 3. *For any non-square $d \in \mathbb{Q}^\times$ the rigid Calabi–Yau threefold \tilde{V}_{24} has a twist $\tilde{V}_{24,d}$ by d . The corresponding involution ι is given by:*

$$\iota : x_1 \mapsto -x_1 \quad (\text{or } x_2 \mapsto -x_2)$$

and all other coordinates fixed with $x_0 \neq 0$.

Proof. Replacing x_1^2 and x_1^4 in the defining equations for \tilde{V}_{24} by dx_1^2 and $d^2x_1^4$, respectively, we get a system of equations that give rise to $\tilde{V}_{24,d}$ isomorphic to \tilde{V}_{24} over $\mathbb{Q}(\sqrt{d})$. The corresponding involution ι is clearly as stated. A holomorphic 3-form on V_{24} is given by

$$\Omega = \frac{dx_3 \wedge dx_4 \wedge dx_5}{8x_1x_2^3 - 8x_1^3x_2}$$

and under the involution $x_1 \mapsto -x_1$, Ω is mapped to $-\Omega$. \square

2.6. The rigid Calabi–Yau threefold of van Geemen and Nygaard. Another interesting example is the case of the rigid Calabi–Yau threefold of van Geemen and Nygaard. In [7], van Geemen and Nygaard gave an example of a rigid Calabi–Yau threefold defined over \mathbb{Q} : Let $Y \subset \mathbb{P}^7$ be the complete intersection of the four quadrics:

$$\begin{aligned} y_0^2 &= x_0^2 + x_1^2 + x_2^2 + x_3^2 \\ y_1^2 &= x_0^2 - x_1^2 + x_2^2 - x_3^2 \\ y_2^2 &= x_0^2 + x_1^2 - x_2^2 - x_3^2 \\ y_3^2 &= x_0^2 - x_1^2 - x_2^2 + x_3^2. \end{aligned}$$

The variety Y has 96 isolated singularities, which are ordinary double points. Let X be a (small) blow-up of Y along its singular locus. (A recent article of Freitag and Salvati-Manni [8] asserts that Y admits a resolution that is a projective Calabi–Yau threefold, X .) Then X is a rigid Calabi–Yau threefold over \mathbb{Q} . Its attached newform is the unique newform of weight 4 on $\Gamma_0(8)$, cf. [7], Theorem 2.4. Notice that there is a misprint in the equations on p. 56 of that paper: In the second equation x_3^2 should occur with a minus sign as above rather than a plus sign as in [7], p. 56. This is evident from the theta relations on p. 54 of [7].

Again, we instantly see the existence of twists of X via replacing x_0^2 by dx_0^2 in the above equations. Thus:

Proposition 4. *For any non-square $d \in \mathbb{Q}^\times$ the above rigid Calabi–Yau threefold X has a twist X_d by d . The corresponding involution ι is given by*

$$x_0 \mapsto -x_0$$

and all other coordinates fixed.

Proof. The only thing we need to check is whether a holomorphic 3-form is sent by ι to $-\Omega$. Let

$$f_0 := y_0^2 - (x_0^2 + x_1^2 + x_2^2 + x_3^2),$$

and similarly, define f_1, f_2 and f_3 by the second, third and the fourth equation, respectively. Then Ω may be given by

$$\Omega = \frac{dx_0 \wedge dx_2 \wedge dx_3}{D}$$

where

$$D = \det \left(\frac{\partial f_i}{\partial y_j} \right)_{0 \leq i, j \leq 3} = 2^4 y_0 y_1 y_2 y_3.$$

Thus the involution given by $\iota : x_0 \mapsto -x_0$ and fixing all other coordinates will send Ω to $-\Omega$. \square

3. REMARKS ON THE LEVELS OF TWISTS

Suppose that $d \in \mathbb{Z}$ is squarefree and suppose that f is a newform of level N . One may ask about the level of the twisted newform f_d . Viewed from the Galois representation side, this amounts to asking for the conductor of $\rho \otimes \chi$ where ρ has conductor N and χ is a (quadratic) character of conductor D , say (so D divides $4d$ in the above setup).

As is well-known, the answer is ND^2 if $(N, D) = 1$. (This can be proved either via basic theory of conductors, or, alternatively, more directly via the theory of modular forms.) When $(N, D) > 1$, however, the question has no simple answer: the level of the twisted representation will depend heavily on the behavior of the representation

ρ at inertia groups over common prime divisors of N and D . Nevertheless, see [10], [11] for the cases where ρ has a ‘small’ image.

For the concrete examples of twisting that we have discussed in this paper, a more modest question can be asked, namely whether the newform that we start with is ‘twist minimal’ or not, i.e., whether it has the lowest level among all of its quadratic twists. This question can be easily answered with a little computation.

Let us for example consider the case of the Schoen quintic. By Proposition 5.3 of [17] (and its proof) one has that the attached newform f is of weight 4 on $\Gamma_0(25)$ given as an explicit linear combination of certain η -products. From this explicit description of f one computes that the coefficient of q^2 in its q -expansion is -84 .

On the other hand, there is a unique newform f_0 of weight 4 on $\Gamma_0(5)$, namely $f_0(z) := \eta(z)^4 \eta(5z)^4$. We compute that the coefficient of q^2 in the q -expansion of f_0 is -4 .

Since $-4 \neq \pm(-84)$ we can deduce that f_0 is not a twist of f and hence that f is twist minimal. In particular, the unique newform f_0 of level 4 is not the form attached to a twist of the Schoen quintic. Is it attached to any Calabi–Yau threefold?

4. FINAL REMARKS

4.1. As we implied in the introduction, the main contribution of this paper is to put focus on the question whether any rigid Calabi–Yau threefold over \mathbb{Q} has a twist by d for any non-square $d \in \mathbb{Q}$. In contrast with the classical situation involving elliptic curves, the question for rigid Calabi–Yau threefolds over \mathbb{Q} seems genuinely more difficult. One difference is that one does not know in general the automorphism group of a rigid Calabi–Yau threefold over \mathbb{Q} . Another point is the poor understanding of the conductor of a rigid Calabi–Yau threefold over \mathbb{Q} , i.e., the level of its associated newform.

For many other cases than the ones we have considered here, the existence of quadratic twists of a given Calabi–Yau threefold over \mathbb{Q} can be shown along the same lines as above, i.e., inspection of the defining equation(s) combined with an application of Theorem 1. For instance, one can try to show the existence of quadratic twists for many rigid Calabi–Yau threefolds over \mathbb{Q} discussed in Meyer [13].

However, in some cases the question does not seem as easy. For instance, does the rigid Calabi–Yau threefold of Hirzebruch (Theorem 5.11 in Yui [21]) admit quadratic twists? Let X_0 be the quintic threefold defined over \mathbb{Q} by the equation $F(x, y) - F(u, w) = 0$ where

$$F(x, y) = \left(x + \frac{1}{2}\right) \left(y^4 - y^2(2x^2 - 2x + 1) + \frac{1}{5}(x^2 + x - 1)^2\right).$$

Then X_0 has 126 nodes (ordinary double points) as only singularities. Let X be the blow up of X_0 along its singular locus. Then X is a rigid Calabi–Yau threefold defined over \mathbb{Q} with the Euler characteristic 306. The map sending y to $-y$ gives rise to an involution on X and this raises the obvious question of whether this induces a non-trivial involution on H^3 so that we get a quadratic twist of X by replacing y^2 by dy^2 .

4.2. Does there exist a rigid Calabi–Yau threefold X defined over \mathbb{Q} and a non-square rational d such that X does not have a quadratic twist X_d by d ?

Perhaps, in order to approach this question, one needs to loosen the definition of ‘quadratic twist by d ’ so that the existence of X_d becomes *equivalent to* (rather than just implying) the existence of a rigid Calabi–Yau threefold over \mathbb{Q} whose attached ℓ -adic Galois representation (for some prime ℓ) is the twist by the quadratic character of $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ of the ℓ -adic representation attached to the original threefold. Maybe this is possible by considering, more generally, algebraic correspondences rather than isomorphisms in the setting of Theorem 1.

Calabi–Yau threefolds, even rigid ones, in general, may not have involutions; even when they do, it might be rather difficult to find one. So geometric realization of modular forms of weight 4 on some $\Gamma_0(N)$ with integral Fourier coefficients along our proposed approach may in fact not be overly promising. But this remark once again raises the question of when the kind of twisting that we have discussed in this paper is possible.

4.3. We should include here a description of the fixed point set of the involution ι acting on a rigid Calabi–Yau threefold (though we did not make use of it in the examples.) The following observation is due to B. van Geemen.

Let X be a Calabi–Yau threefold over \mathbb{Q} . Let ι be an involution acting on X . Then the fixed point set of ι on X is determined as follows.

Suppose that p is a fixed point of ι , then in suitable local coordinates $z_i, i = 1, 2, 3$, $z_i(p) = 0$, and

$$\iota(z_1, z_2, z_3) = (e_1 z_1, e_2 z_2, e_3 z_3) \quad \text{where } e_i = \pm 1,$$

and the dimension of the fixed point set is thus the number of e_i ’s which are equal to $+1$ (so if $\iota \neq 1$, then at least one e_i must be -1 .)

If Ω is the nowhere vanishing holomorphic 3-form in $H^{3,0}(X)$, then in these coordinates,

$$\Omega = f(z_1, z_2, z_3) dz_1 \wedge dz_2 \wedge dz_3 \quad \text{and } f(0, 0, 0) \neq 0.$$

But $f(0, 0, 0) \neq 0$ forces that

$$f(e_1 z_1, e_2 z_2, e_3 z_3) = +f(z_1, z_2, z_3),$$

and hence

$$\iota^* \Omega = e_1 e_2 e_3 \Omega.$$

From this we see that if the fixed point locus consists of isolated points and divisors, then all or exactly one of the e_i are -1 , else two of the $e_i = +1$ and the one -1 .

5. APPENDIX

Proof of Lemma 1. Permuting variables if necessary we may, and will, assume $i_0 = 0$ and $I = \{1, \dots, k\}$. We put $D := D_I$ with this particular I .

Let us first recall some general facts about the Griffiths residue map: Suppose that V is a smooth projective variety of dimension n and that W is a smooth codimension 1 subvariety, or, more generally, a normal crossings divisor. The Griffiths residue map (see for instance the proof of Prop. 8.32 in [19]) is a surjective homomorphism

$$\text{Res} : \Omega_V^n(\log W) \rightarrow \Omega_W^{n-1}$$

defined as follows: If $(U, (z_1, \dots, z_n))$ is a complex chart of V such that W is given locally by the equation $z_1 = 0$ then $\Omega_V^n(\log W)|_U$ is the free sheaf of \mathcal{O}_U -modules generated by

$$\alpha := \frac{dz_1}{z_1} \wedge dz_2 \wedge \dots \wedge dz_n,$$

and Res is then defined locally by

$$\text{Res}(g\alpha) := (gdz_2 \wedge \dots \wedge dz_n)_{U \cap W}.$$

Notice that $(U \cap W, (z_1, \dots, z_{n-1})_U)$ is a complex chart of W .

Now, for the proof of the Lemma, let us first consider the case $k = 1$ where we specialize the above to the situation $V = \mathbb{P}^n$ and $W = Y$ the hypersurface given by the equation $f_1 = 0$. Consider the open set U_0 where $x_0 \neq 0$, let $z_i := x_i/x_0$ on U_0 , and let $F := f_1/x_0^t$ where t is the degree of f_1 . Then $\omega_n := dz_1 \wedge \dots \wedge dz_n$ is a generator of $\Omega_{U_0}^n$.

The open subset U'_0 of U_0 where $\partial F/\partial z_1 \neq 0$ coincides with the open subset where $\partial f_1/\partial x_1 \neq 0$. On U'_0 we have local coordinates F, z_2, \dots, z_n . Since

$$dF = \sum_{i=1}^n (\partial F/\partial z_i) dz_i$$

we find

$$dF \wedge dz_2 \wedge \dots \wedge dz_n = \left(\frac{\partial F}{\partial z_1} \right) \omega_n$$

so that

$$\text{Res}\left(\frac{\omega_n}{F}\right)_{|U'_0} = \text{Res}\left(\frac{dF \wedge dz_2 \wedge \dots \wedge dz_n}{F \left(\frac{\partial F}{\partial z_1}\right)}\right)_{|U'_0} = \left(\frac{dz_2 \wedge \dots \wedge dz_n}{\left(\frac{\partial F}{\partial z_1}\right)}\right)_{|U'_0}$$

which coincides up to a power of x_0 with

$$\frac{dx_2 \wedge \dots \wedge dx_n}{\left(\frac{\partial f_1}{\partial x_1}\right)}$$

on U'_0 .

Since the residue is holomorphic on all of Y , this differential form on U'_0 will extend holomorphically to all of Y .

For the general case, one can argue inductively with respect to k : We see Y as the end of a chain $Y := Y_k \subseteq \dots \subseteq Y_1 \subseteq \mathbb{P}^n$ where each Y_i is a codimension 1 subvariety of the Y_{i-1} defined by the equation $f_i = 0$. Dividing by suitable powers of x_0 to define F_i from f_i as above and retaining local coordinates $z_i := x_i/x_0$ for $i > 0$, the conclusion is now that we get a holomorphic 3-form on U_0 (where $x_0 \neq 0$) by taking the residue of the form $\omega_n/(F_1 \cdots F_k)$.

If we define $\mathcal{D} := \det \left(\frac{\partial F_i}{\partial z_j} \right)_{1 \leq i, j \leq k}$ then

$$dF_1 \wedge \dots \wedge dF_k \wedge dz_{k+1} \wedge \dots \wedge dz_n = \mathcal{D} \cdot dz_1 \wedge \dots \wedge dz_n$$

as $dF_j = \sum_{i=1}^n (\partial F_j/\partial z_i) dz_i$, and by the definition of the determinant and alternating property of wedge products. Redefining U'_0 as the open subset of U_0 where $\mathcal{D} \neq 0$ then U'_0 coincides with the open subset of U_0 where $D \neq 0$ as D differs from \mathcal{D} by a power of x_0 .

Thus, on U'_0 we can compute the above residue:

$$\begin{aligned} \text{Res}\left(\frac{\omega_n}{F_1 \cdots F_k}\right)_{|U'_0} &= \text{Res}\left(\frac{dF_1 \wedge \cdots \wedge dF_k \wedge dz_{k+1} \wedge \cdots \wedge dz_n}{F_1 \cdots F_k \mathcal{D}}\right)_{|U'_0} \\ &= \left(\frac{dz_{k+1} \wedge \cdots \wedge dz_n}{\mathcal{D}}\right)_{|U'_0} \end{aligned}$$

which coincides with

$$\frac{dx_{k+1} \wedge \cdots \wedge dx_n}{D}$$

up to a power of x_0 on U'_0 .

Again this form extends to all of Y for the same reasons as in the case $k = 1$.

Now suppose that Y has a crepant resolution X that is Calabi-Yau variety of dimension $\dim X \leq 3$. There is then a surjective map $\Omega_X^3 \rightarrow \Omega_Y^3$. Hence the holomorphic 3-form on Y that we constructed above on Y extends to a holomorphic 3-form on X . \square

6. ACKNOWLEDGMENTS

We would like to thank Bert van Geemen, Ken-Ichiro Kimura, James D. Lewis, and Matthias Schütt for helpful discussions on this topic.

We would also like to thank the referee for going meticulously through the manuscript and pointing out a number of inaccuracies in the previous version.

The first author thanks Queen's University for its hospitality when this work was done and Colby College for research funding. The second author thanks The Danish Council for Independent Research for support. The third author thanks the Natural Sciences and Engineering Research Council of Canada (NSERC) for their support.

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